Can a Quantum Nondemolition Measurement Improve the Sensitivity of an Atomic Magnetometer?

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We consider the limitations due to noise (e.g., quantum projection noise and photon shot-noise) on the sensitivity of an idealized atomic magnetometer that utilizes spin squeezing induced by a continuous quantum nondemolition measurement. Such a magnetometer measures spin precession of N atomic spins by detecting optical rotation of far-detuned light. We show that for very short measurement times, the optimal sensitivity scales as $N^{-3/4}$; if strongly squeezed probe light is used, the Heisenberg limit of N^{-1} scaling can be achieved. However, if the measurement time exceeds $\tau_{\rm rel}/N^{1/2}$ in the former case, or $\tau_{\rm rel}/N$ in the latter, where $\tau_{\rm rel}$ is the spin relaxation time, the scaling becomes $N^{-1/2}$, as for a standard shot-noise-limited magnetometer.

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Recently there has been considerable interest in improving the sensitivity of precision measurements (including those associated with atomic clocks, see, for example, Ref. [1] and references therein) using techniques associated with quantum entanglement and spin squeezing. One example is recent work [2] reporting an atomic magnetometer with noise below the shot-noise limit. This experiment utilized a quantum nondemolition (QND) measurement of atomic spins—an optical-rotation measurement with off-resonant light—to achieve spin squeezing (see, for example, Refs. [3–5], and references therein). The purpose of this technique is to reduce the influence of the quantum-mechanical spin-projection noise.

Here we consider limitations of some commonly employed QND techniques. For concreteness, we analyze an idealized magnetometer and determine the scaling of the sensitivity with various key parameters of the system (e.g., the number of atoms N and the measurement time τ). In this magnetometer scheme (Fig. 1), a circularly polarized pump beam orients N paramagnetic atoms along $\hat{\mathbf{x}}$.

When the pump beam is turned off, the atomic spins precess around the direction of the magnetic field to be measured, assumed here to be along $\hat{\mathbf{y}}$. The spin precession is detected using optical rotation of a far-detuned ($|\Delta| \gg \Gamma_0$, where Γ_0 is the natural transition width and Δ is the frequency detuning from optical resonance), linearly polarized probe beam propagating along $\hat{\mathbf{z}}$, with cross section (of area A) assumed to match that of the atomic sample.

According to general principles of quantum mechanics, a measurement perturbs the quantum state of the system under observation. However, if one is not attempting to extract the complete information about the system, it is possible to set up a QND measurement that will not strongly affect the quantity one is trying to determine (see, for example, Ref. [6]). Specifically, in the case considered here, the orientation of the atomic spins in a given direction is measured via optical rotation of the probe beam. If a photon is absorbed from the probe-light beam, the atom is excited from the state one is attempting to measure and the orientation of the system is altered. The photon-absorption probability scales with detuning as $1/\Delta^2$, while optical rotation due to the imbalance of the number of atoms oriented along and opposite to the light-propagation direction scales as $1/\Delta$. Thus, an approximation to a QND measurement of orientation is realized by simply tuning the light sufficiently far away from resonance. However, the residual absorption turns out to be important in optimizing the measurement, as discussed below.

We assume that the pump beam prepares the N paramagnetic atoms with all spins polarized in the $\hat{\mathbf{x}}$ direction. Without loss of generality, we can assume that the magnetic field to be detected is arbitrarily small. The measurement of the optical rotation is carried out over

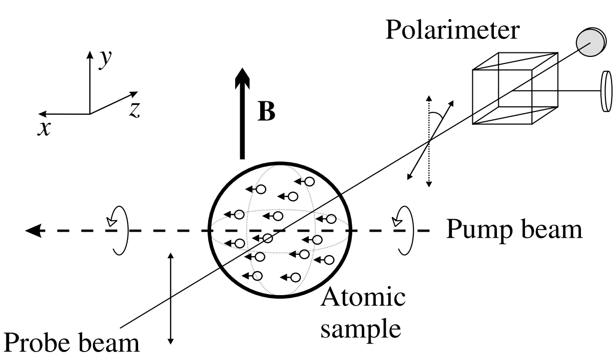

FIG. 1. Schematic diagram of a typical atomic magnetometer apparatus [2,9] of the sort considered here.

a short (as defined below) time τ. In order to make our argument as transparent as possible and to simplify the mathematical expressions, in the following we neglect numerical constants of order unity and set $\hbar = c = 1$.

First, we recall the principle of the magnetometer's operation (see, for example, Ref. [7] for a general discussion). The polarized atoms undergo Larmor precession in the magnetic field [8], tipping their polarization direction from the initial $\hat{\mathbf{x}}$ direction towards $\hat{\mathbf{z}}$ by an angle $g\mu B\tau$ during the measurement time τ. Here g is the Landé factor and μ is the Bohr magneton. The angle of optical rotation induced by the excess of atoms with spin projection along $\hat{\mathbf{z}}$ can be written as

$$\varphi = g\mu B\tau \frac{l}{l_0}\frac{\Gamma_0}{\Delta}. \tag{1}$$

Here l is the length of the sample in the direction of the light propagation, l_0 is the on-resonance unsaturated absorption length, and the expression assumes far-detuned light and a weak magnetic field.

Consider now two fundamental sources of noise that limit how well one can determine B from Eq. (1) (we assume that the noise in the magnetic field is negligible). First, there is photon shot noise in the optical polarimeter

$$\delta\varphi_{\rm ph} = \frac{1}{\sqrt{N_{\rm ph}}}, \tag{2}$$

where $N_{\rm ph}$ is the total number of photons used in the measurement. From Eq. (1), this noise translates into the magnetic field detection limit

$$\delta B_{\rm ph} = \frac{1}{g\mu\tau}\frac{1}{N\sqrt{N_{\rm ph}}}\frac{\Delta}{\Gamma_0}\frac{A}{\lambda^2}, \tag{3}$$

where we have written the resonant absorption length l_0 (which can be thought of as a mean free path for a resonant photon) as

$$l_0 = \frac{1}{n\lambda^2} = \frac{Al}{N\lambda^2}, \tag{4}$$

where n is the number density of the atoms and λ is the light wavelength.

The other source of noise is related to the fact that, even though the probe light is far-detuned from resonance, it still excites a number N_e of atoms, given by the product of the resonant excitation rate, a scaling factor taking into account the large light detuning, and $N\tau$:

$$N_e = \frac{d^2E^2}{\Gamma_0}\left(\frac{\Gamma_0}{\Delta}\right)^2 N\tau = NN_{\rm ph}\left(\frac{\Gamma_0}{\Delta}\right)^2\frac{\lambda^2}{A}, \tag{5}$$

where d is the dipole moment of the probe transition and E is the amplitude of the probe-light field. Here we have used $d^2 = \Gamma_0\lambda^3$ and

$$E^2 = \frac{\Phi}{\lambda} = \frac{N_{\rm ph}}{\lambda A\tau}, \tag{6}$$

where $\Phi = N_{\rm ph}/(A\tau)$ is the photon flux. Such excitation results in a random imbalance of order $\sqrt{N_e}$ between the number of atoms with positive and negative spin projection along $\hat{\mathbf{z}}$. This leads to optical rotation by a random angle

$$\delta\varphi_{\rm at} \sim \frac{\sqrt{N_e}}{N}\frac{l}{l_0}\frac{\Gamma_0}{\Delta}, \tag{7}$$

and a corresponding uncertainty

$$\delta B_{\rm at} = \frac{1}{g\mu\tau}\frac{\sqrt{N_{\rm ph}}}{\sqrt{N}}\frac{\Gamma_0}{\Delta}\left(\frac{\lambda^2}{A}\right)^{1/2}. \tag{8}$$

It is important to emphasize that the uncertainty in magnetic-field determination described by Eq. (8) arises solely due to optical pumping induced by the *probe* beam during the measurement time τ. As shown in Refs. [2,5], the projection noise due to the initial spin preparation can be eliminated by use of a proper measurement procedure.

We see that the two contributions to uncertainty in the magnetic-field determination—one associated with the polarimeter photon noise, the other associated with reorientation of atoms by the probe light—have opposite dependences on $N_{\rm ph}$. We can find the optimum number of photons by minimizing the overall uncertainty. Differentiating the sum in quadrature of the contributions of Eqs. (3) and (8) by $N_{\rm ph}$ and setting the derivative to zero, we find the optimal value

$$N_{\rm ph}^{\rm opt} = \frac{1}{N^{1/2}}\left(\frac{\Delta}{\Gamma_0}\right)^2\left(\frac{A}{\lambda^2}\right)^{3/2}, \tag{9}$$

for which the photon and atomic noise contributions are the same. The resultant overall uncertainty in the determination of the magnetic field in a single measurement of length τ is

$$\delta B = \frac{1}{g\mu\tau}\frac{1}{N^{3/4}}\left(\frac{A}{\lambda^2}\right)^{1/4}. \tag{10}$$

Note that the transition line width and frequency detuning have dropped out of the optimized result (10). Equation (10) shows that the sensitivity to the magnetic-field scales as $N^{-3/4}$, better than the scaling $N^{-1/2}$ for a usual shot-noise-limited measurement [7,9], but still short of the result N^{-1} obtained in the Heisenberg limit. The factor $(A/\lambda^2)^{1/4}$ indicates that, given a total number of atoms N, it is beneficial to compress their dimensions down to the wavelength of the light, maximizing the optical-rotation angle. This, however, may be difficult to achieve experimentally, and may also lead to cooperative effects, not considered here, in the light-atom interaction.

It is interesting to note that with an optimized measurement the number of atoms that undergo optical pumping during the measurement time τ is [using Eq. (9)]

$$\frac{d^2E^2}{\Delta^2}\Gamma_0\tau N = \lambda^{-1}A^{1/2}\sqrt{N}. \tag{11}$$

Next, we consider the possibility of improving magnetometric sensitivity by employing strongly squeezed probe light [6]. In this case, the photon noise contribution approaches $1/N_{\rm ph}$ [cf. Eq. (2)] when 100%-efficient photodetection is assumed; the minimization of the uncertainty in the magnetic-field determination leads to the optimal number of photons

$$N_{\rm ph}^{\rm opt} = \frac{1}{N^{1/3}}\left(\frac{\Delta}{\Gamma_0}\right)^{4/3}\frac{A}{\lambda^2}, \tag{12}$$

and uncertainty in magnetic-field detection of

$$\delta B = \frac{1}{g\mu\tau}\frac{1}{N^{2/3}}\left(\frac{\Gamma_0}{\Delta}\right)^{1/3}. \tag{13}$$

In contrast to the case of unsqueezed light [Eq. (10)], the detuning Δ has not canceled, while the area A has. This seems to be an improvement on both fronts. To obtain the greatest sensitivity, the atomic sample no longer needs to be compressed to the scale of the light wavelength. Also, it would appear that δB can be decreased without limit by increasing the detuning. However, there is a fundamental limit to the sensitivity that can be derived from the Heisenberg uncertainty principle:

$$\delta B_H = \frac{1}{g\mu\tau}\frac{1}{N}. \tag{14}$$

Equating (13) and (14), we find that the Heisenberg limit is achieved when $\Delta = N\Gamma_0$. Putting this value of the detuning into Eqs. (5) and (12), we see that the optimal number of photons is $N_{\rm ph}^{\rm opt} = NA/\lambda^2$ and the number of atoms optically pumped during the measurement is of the order of unity. Indeed, since the Heisenberg limit is reached when the change of one atomic spin due to the magnetic field can be measured, a greater number of spins must not be disturbed by the light.

In order to obtain the greatest sensitivity to magnetic fields, it is advantageous to make a single QND measurement over as long a time as possible. Up until now, we have ignored the ground-state spin relaxation (with rate $\Gamma_{\rm rel}$), as we have assumed τ sufficiently short. For longer measurement times, the approximation that the spins reorient only due to optical pumping by the probe beam will fail. For the case of a measurement using squeezed light, when the number of spins ($N\Gamma_{\rm rel}\tau$) that flip due to ground-state relaxation becomes comparable to unity, uncertainty due to relaxation begins to dominate the atomic noise. The additional noise in the optical-rotation angle due to the relaxation of $N\Gamma_{\rm rel}\tau$ atoms during the measurement is given, analogously to Eq. (7), by

$$\delta\varphi_{\rm rel} \sim \frac{\sqrt{N\Gamma_{\rm rel}\tau}}{N}\frac{l}{l_0}\frac{\Gamma_0}{\Delta}. \tag{15}$$

The corresponding noise in the magnetic-field determination is given by

$$\delta B_{\rm rel} = \frac{\sqrt{\Gamma_{\rm rel}}}{g\mu\sqrt{N\tau}}. \tag{16}$$

Thus it is evident that if the measurement is performed over a time $\tau \gg (N\Gamma_{\rm rel})^{-1}$ any advantage in sensitivity due to the QND measurement is lost since the noise scales as $N^{-1/2}$. For a total measurement time $T \gg (N\Gamma_{\rm rel})^{-1}$, one can perform $T/\tau = N\Gamma_{\rm rel}T$ independent Heisenberg-limited measurements of the magnetic field, each of duration $\tau = (N\Gamma_{\rm rel})^{-1}$ and sensitivity $\Gamma_{\rm rel}/(g\mu)$ [Eq. (14)]. The total uncertainty in the measurement improves as the square root of the number of such independent measurements. Thus the sensitivity achieved during the measurement time is given by

$$\delta B = \frac{\sqrt{\Gamma_{\rm rel}}}{g\mu\sqrt{NT}}, \tag{17}$$

which is the same as the sensitivity of a conventional shot-noise-limited magnetometer.

A similar conclusion is also reached for the case of unsqueezed probe light. Here, the maximal measurement time during which no relaxation events that would spoil the sensitivity can occur is

$$\tau = \frac{1}{\sqrt{N}}\frac{1}{\Gamma_{\rm rel}}\left(\frac{A}{\lambda^2}\right)^{1/2}, \tag{18}$$

which once again leads us to the result (17). A similar result was obtained in the context of frequency measurements in the presence of decoherence [10], where it was shown that optimal measurements with maximally entangled states offer no improvement over standard spectroscopic techniques.

The preceding analysis suggests the general result that, while it is possible to perform measurements that go beyond the shot-noise limits for very short times, the inevitable presence of ground-state relaxation means that the most sensitive measurements—requiring longer measurement times—will have the usual shot-noise scaling of the sensitivity. As a numerical example, Heisenberg-limited measurements for $N = 10^{11}$ and $\Gamma_{\rm rel} = 100$ Hz (parameters comparable to those used in Ref. [2]) must be shorter than 10^{-13} s.

So far, we have considered two fundamental limits to the magnetometric sensitivity: photon shot-noise and optical pumping by the probe light. In addition to these sources of noise, the probe beam also contributes noise due to quantum fluctuations of its polarization. This leads to a differential ac Stark shift of the ground-state magnetic sublevels. Although this effect does not change the scaling of the magnetometric precision, as we show below, it is important to account for such noise when considering the Heisenberg uncertainty relations for the atomic spins.

Although the probe beam is nominally linearly polarized, vacuum fluctuations in the orthogonal polarization can create a small admixture of random circular polarization. The magnitude of the quantum fluctuations of the probe polarization can be found using the ellipticity operator $\hat{\epsilon}$ for nominally y-polarized light [11]

$$\hat{\epsilon} = \frac{\mathcal{E}_0}{2iE}(\hat{a}_x - \hat{a}_x^\dagger), \tag{19}$$

where $\mathcal{E}_0$ is the characteristic amplitude of unsqueezed vacuum fluctuations (see, for example, Ref. [6]) and $\hat{a}_x$, $\hat{a}_x^\dagger$ are the annihilation and creation operators for x-polarized photons at the same frequency as the probe. Assuming that the x-polarized field is the unsqueezed vacuum, we find for the quantum fluctuations of the probe beam's ellipticity (details of the calculation are presented in Ref. [12]):

$$\delta\epsilon = \sqrt{\langle\hat{\epsilon}^2\rangle - \langle\hat{\epsilon}\rangle^2} = \frac{1}{\sqrt{N_{\rm ph}}}. \tag{20}$$

The magnitude of the differential ac Stark shift of the ground-state magnetic sublevels, $\delta\Delta_{\rm ac}$, due to the fluctuations of the probe polarization is

$$\delta\Delta_{\rm ac} = \frac{d^2E^2}{\Delta}\delta\epsilon. \tag{21}$$

This causes the atomic polarization vector to precess by a random angle in the x-y plane (Fig. 1). (This small-angle precession in the x-y plane does not affect the magnetometric sensitivity.) After time τ this random angle has a magnitude $\alpha_{x-y} = \tau\delta\Delta_{\rm ac}$. After substitutions from Eqs. (6), (9), (20), and (21), the following expression is obtained:

$$\alpha_{x-y} = \frac{1}{N^{1/4}}\left(\frac{A}{\lambda^2}\right)^{-1/4}. \tag{22}$$

This rotation of the atomic polarization vector in the x-y plane ensures that the measurement uncertainties obey the Heisenberg uncertainty relation

$$\delta J_z \delta J_y \geq J_x. \tag{23}$$

Let us verify this. From Eq. (10) we obtain

$$\delta J_z = N\tau g\mu\delta B = N^{1/4}(A/\lambda^2)^{1/4}. \tag{24}$$

Using Eq. (22),

$$\delta J_y = N\alpha_{x-y} = N^{3/4}(A/\lambda^2)^{-1/4}. \tag{25}$$

To first order, $J_x = N$, so the uncertainty relation (23) is satisfied.

In the case of squeezed probe light, the random admixture of circular polarization due to quantum fluctuations is in fact $\delta\epsilon \sim 1$; this result can be derived in the same manner as Eq. (20), except here one uses a squeezed vacuum state (ensuring the photon noise in the probe optical-rotation measurement at the $1/N_{\rm ph}$ level) for the x-polarized field. Repeating the ac Stark effect calculation (with the factor A/λ^2 set to unity) gives the angle of rotation in the x-y plane $\alpha_{x-y} \sim 1$, which means that $\delta J_y = J_x = N$. But at the Heisenberg limit $\delta J_z = 1$, so once again the uncertainty relation (23) holds.

In conclusion, we have investigated fundamental sources of noise present in an idealized atomic magnetometer based on quantum nondemolition techniques. We find that such an approach can improve the sensitivity of magnetometric measurements beyond the shot-noise-limit over time scales much shorter than the relevant spin-relaxation time divided by an appropriate power of the number of atoms, depending on the degree of squeezing of the probe light. However, for longer time scales, even if squeezed probe light is employed, QND techniques offer no significant improvement in the sensitivity of magnetic-field measurements.

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